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Note on the Bayad Reciprocity Law in an Imaginary Quadratic Number Field II

by

Heima HAYASHI[1]

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Abstract

This paper is a sequel to our previous paper of the same title [6]. Using the results in Bayad-Ayala [2], the original Bayad function is expressed with use of the Klein function, and all our results in [6] can be rewritten in a renewed version (Theorems 1-2 and 4-1). In particular, we are deeply concerned with the quantities $\xi_\Omega(\alpha)$ and $A_\alpha^{(\beta)}$ defined in [6] respectively with relation to the product formula of Bayad function and the law of quadratic reciprocity in an imaginary quadratic number field (Theorems 2-3 and 4-1). It is remarkable that our renewed formula of quadratic reciprocity law has the quite similar form to one of the formula of Hajir-Villegas [5], and this fact provides us another interesting problem (Theorems 4-2, 4-3 and 4-4).

1. Terminologies and reformulation of a result on the Bayad function.

Let $\boldsymbol{C}, \boldsymbol{R}$ and $\boldsymbol{Z}$ be respectively the field of complex and real numbers and the ring of rational integers. By a $\boldsymbol{C}$-lattice we mean a free $\boldsymbol{Z}$-module of rank 2 which spans $\boldsymbol{C}$ over $\boldsymbol{R}$. For $\boldsymbol{C}$-lattice Ω with $\boldsymbol{Z}$-basis $\{\omega_1, \omega_2\}$ such that $\mathrm{Im}(\omega_1/\omega_2) > 0$,

$$a(\Omega) := \frac{1}{2\,i}\begin{vmatrix} \omega_1 & \omega_2 \\ \overline{\omega}_1 & \overline{\omega}_2 \end{vmatrix} = |\omega_2|^2\,\mathrm{Im}\big(\frac{\omega_1}{\omega_2}\big)$$

is a real positive number, which means the area of fundamental parallelogram of Ω and depends only on Ω. Let E_Ω be a $\boldsymbol{R}$-bilinear form defined by

$$E_\Omega(u,v) := \frac{1}{2\,i\,a(\Omega)}\,(\overline{u}v - u\overline{v}) \qquad \text{for} \quad (u,v) \in \boldsymbol{C}\times\boldsymbol{C}.$$

Then E_Ω is integral valued on $\Omega\times\Omega$ and $E_\Omega(\omega_1,\omega_2) = -1$ for any basis $\{\omega_1, \omega_2\}$ of Ω such that $\mathrm{Im}(\omega_1/\omega_2) > 0$.

Here we summarize briefly about the Klein function $\mathcal{K}_\Omega$, the Jacobi form D_Ω and their fundamental properties, quoting mainly from Bayad-Ayala [2]. For the detail, one should refer also to Kubert [7] and Kubert-Lang [8]. The Klein function $\mathcal{K}_\Omega$ attached to a $\boldsymbol{C}$-lattice Ω is defined by the infinite product

$$\mathcal{K}_\Omega(z) = z\,e^{-\frac{1}{2}z\,\eta(z,\Omega)} \prod_{\omega\in\Omega\setminus\{0\}} \big(1-\frac{z}{\omega}\big)\,e^{\frac{z}{\omega}+\frac{1}{2}\left(\frac{z}{\omega}\right)^2}$$

[1]Department of Mathematics, Faculty of Science.

for any $z \in \boldsymbol{C}$, where $\eta(z, \Omega)$ means the Weierstrass-Legendre eta function attached to Ω. $\mathcal{K}_\Omega$ has the following fundamental properties:
(K1) For $\rho \in \Omega$

$$\mathcal{K}_\Omega(z+\rho) = \chi_\Omega(\rho)\, e(\, E_\Omega(\rho, z)/2\,)\, \mathcal{K}_\Omega(z),$$

where

$$\chi_\Omega(\rho) = \begin{cases} 1 & \text{if } \rho \in 2\Omega, \\ -1 & \text{if } \rho \in \Omega \setminus 2\Omega, \end{cases}$$

and $e(x) = e^{2\pi i x}$ for $x \in \boldsymbol{R}$.
(K2) $\mathcal{K}_\Omega(z)$ is homogeneous of degree 1, that is

$$\mathcal{K}_{\lambda\Omega}(\lambda z) = \lambda \mathcal{K}_\Omega(z) \qquad \text{for} \quad \lambda \in \boldsymbol{C}^\times := \boldsymbol{C} \setminus \{0\}.$$

In particular, $\mathcal{K}_\Omega(-z) = -\mathcal{K}_\Omega(z)$.
(K3) $\mathcal{K}_\Omega(z)$ admits principal part z when z tends to 0, that is

$$\lim_{z \to 0} \frac{\mathcal{K}_\Omega(z)}{z} = 1.$$

Let Ω and Λ be two $\boldsymbol{C}$-lattices such that $\Omega \subset \Lambda$, and $\mathcal{R}$ be any complete system of representatives of Λ/Ω. Then the following product formula holds:

$$\text{(K4)} \qquad \mathcal{K}_\Lambda(z) = e\big(\, E_\Omega(\, z\, ,\, \sum_{\substack{x \in \mathcal{R} \\ x \notin \Omega}} x\,)/2\,\big)\, \mathcal{K}_\Omega(z) \prod_{\substack{x \in \mathcal{R} \\ x \notin \Omega}} \frac{\mathcal{K}_\Omega(z+x)}{\mathcal{K}_\Omega(x)}.$$

The Jacobi form D_Ω associated with $\mathcal{K}_\Omega$ is defined by

$$\text{(1.1)} \qquad D_\Omega(z\,;\, \varphi) = e\big(\, E_\Omega(z\,,\, \varphi)/2\,\big) \frac{\mathcal{K}_\Omega(z+\varphi)}{\mathcal{K}_\Omega(z)\mathcal{K}_\Omega(\varphi)} \qquad \text{for} \quad z,\, \varphi \in \boldsymbol{C} \setminus \Omega.$$

$D_\Omega(z\,;\,\varphi)$ satisfies the following fundamental properties:
(D1) $D_\Omega(z\,;\,\varphi+\rho) = D_\Omega(z\,;\,\varphi)$ for any $\rho \in \Omega$.
(D2) $D_\Omega(z+\rho\,;\,\varphi) = e\big(\, E_\Omega(\rho, \varphi)\,\big) D_\Omega(z\,;\,\varphi)$ for any $\rho \in \Omega$.
(D3) $D_\Omega(z\,;\,\varphi) = e\big(\, E_\Omega(z\,,\,\varphi)\,\big) D_\Omega(\varphi\,;\,z)$.
(D4) $D_\Omega(z\,;\,\varphi)$ is homogeneous of degree -1, that is

$$D_{\lambda\Omega}(\lambda z\,;\,\lambda\varphi) = \lambda^{-1} D_\Omega(z\,;\,\varphi) \qquad \text{for} \quad \lambda \in \boldsymbol{C}^\times.$$

(D5) $D_\Omega(z\ \varphi)$ admits principal part $\frac{1}{z}$ when z tends to 0, that is

$$\lim_{z \to 0} z \cdot D_\Omega(z\,;\,\varphi) = 1.$$

Let Ω, Λ and $\mathcal{R}$ be the same as before. Then the main theorem in [2] asserts the following product formulas:
(D6) For any z and $\varphi \in \boldsymbol{C} \setminus \Lambda$,

$$D_\Lambda(z\,;\,\varphi) = \frac{\mathcal{K}_\Omega(\varphi)^{[\Lambda:\Omega]}}{\mathcal{K}_\Lambda(\varphi)} \prod_{x \in \mathcal{R}} D_\Omega(z+x\,;\,\varphi) e\big(-E_\Omega(x\,,\,\varphi)\big).$$

(D7) For any $z \in \boldsymbol{C} \setminus \Lambda$,

$$\prod_{x \in \mathcal{R}, x \notin \Omega} D_\Omega(z\,;\,x)^{-1} = \frac{\mathcal{K}_\Omega(z)^{[\Lambda:\Omega]}}{\mathcal{K}_\Lambda(z)}.$$

With relation to the Weierstrass $\wp$-function, the following formulas hold:
(D8) For any $z, \varphi \in \boldsymbol{C} \setminus \Omega$, $\wp_\Omega(z) - \wp_\Omega(\varphi) = D_\Omega(z\,;\,\varphi) D_\Omega(z\,;\,-\varphi)$.
(D9) For any $z \in \boldsymbol{C} \setminus \Omega$, $\wp'_\Omega(z) = -2 \prod_{\varphi} D_\Omega(z\,;\,\varphi)$, where φ runs over the set of representatives of $\frac{1}{2}\Omega/\Omega$ such that $\varphi \notin \Omega$.

The Bayad function f_Ω attached to a $\boldsymbol{C}$-lattice Ω with basis $\{\omega_1, \omega_2\}$ is originally defined by

$$f_\Omega(z) = C\,\frac{\wp_\Omega(z) - \wp_\Omega\big(\frac{\omega_1+\omega_2}{2}\big)}{\wp'_\Omega(z)}$$

with the constant C such that

$$C^2 = \frac{2\wp''_\Omega\big(\frac{\omega_2}{2}\big)}{\wp_\Omega\big(\frac{\omega_2}{2}\big) - \wp_\Omega\big(\frac{\omega_1+\omega_2}{2}\big)}$$

(see [1] and [6]). Of course this definition of f_Ω depends on the choice of the basis $\{\omega_1, \omega_2\}$ of Ω. By the formulas (D8) and (D9), $f_\Omega(z)$ can be reformed as follow:

$$\begin{aligned} f_\Omega(z) &= -\frac{C}{2}\,\frac{D_\Omega\big(z\,;\,\frac{\omega_1+\omega_2}{2}\big)}{D_\Omega\big(z\,;\,\frac{\omega_1}{2}\big) D_\Omega\big(z\,;\,\frac{\omega_3}{2}\big)} \\ &= -\frac{C}{2}\,\frac{\mathcal{K}_\Omega\big(\frac{\omega_1}{2}\big)\mathcal{K}_\Omega\big(\frac{\omega_2}{2}\big)}{\mathcal{K}_\Omega\big(\frac{\omega_1+\omega_2}{2}\big)} \times \frac{\mathcal{K}_\Omega\big(z+\frac{\omega_1+\omega_2}{2}\big)\,\mathcal{K}_\Omega(z)}{\mathcal{K}_\Omega\big(z+\frac{\omega_1}{2}\big)\mathcal{K}_\Omega\big(z+\frac{\omega_2}{2}\big)} \\ &:= C_1\,\frac{\mathcal{K}_\Omega\big(z+\frac{\omega_1+\omega_2}{2}\big)\,\mathcal{K}_\Omega(z)}{\mathcal{K}_\Omega\big(z+\frac{\omega_1}{2}\big)\mathcal{K}_\Omega\big(z+\frac{\omega_2}{2}\big)}. \end{aligned}$$

A simple calculation (using (D8) and (D9)) shows that $C_1 = \pm e\big(\frac{1}{8}E_\Omega(\omega_2\,,\,\omega_1)\big)$. Hence we may adopt

$$(1.2)\qquad f_\Omega(z) = e\Big(\frac{1}{8}E_\Omega(\omega_2\,,\,\omega_1)\Big)\,\frac{\mathcal{K}_\Omega\big(z+\dfrac{\omega_1+\omega_2}{2}\big)\,\mathcal{K}_\Omega(z)}{\mathcal{K}_\Omega\big(z+\dfrac{\omega_1}{2}\big)\mathcal{K}_\Omega\big(z+\dfrac{\omega_2}{2}\big)},$$

as the definition of Bayad function. Of course this definition also depends on the choice of basis $\{\omega_1,\omega_2\}$ of Ω. f_Ω is an Ω-elliptic function and its divisor on $\boldsymbol{C}/\Omega$ is

$$(\,f_\Omega\,) \;=\; (\frac{\omega_1+\omega_2}{2}\,) \;+\; (\,0\,) \;-\; (\frac{\omega_1}{2}) \;-\; (\frac{\omega_2}{2}).$$

The following lemma is immediate from the definition (1.1) and the formula (K1).

Lemma 1-1([6, Lemma 1-1]). *Under the above notations, we have*

$$(1)\;\; f_\Omega(z)\cdot f_\Omega(z+\frac{\omega_1}{2}) = 1, \qquad (2)\;\; f_\Omega(z)\cdot f_\Omega(z+\frac{\omega_2}{2}) = -1.$$

Hereafter let k be an imaginary quadratic number field and $\mathfrak{o}_k$ be the ring of integers in k. We mean by $J^*(2)$ the set $\{\,\alpha\in\mathfrak{o}_k \mid (\alpha\,,\,2\mathfrak{o}_k)=1\,\}$. For a fixed $\mathfrak{o}_k$-ideal Ω and an element α in $\mathfrak{o}_k$, we define

$$\mathrm{Ker}(\alpha) = \mathrm{Ker}_\Omega(\alpha) := \{\; x\in \boldsymbol{C}/\Omega \;\mid\; \alpha x = 0\;\}.$$

We call the elements in $\mathrm{Ker}(\alpha)$ α-division points of $\boldsymbol{C}/\Omega$. In particular, $x\in\mathrm{Ker}(\alpha)$ is called a primitive α-division point of $\boldsymbol{C}/\Omega$ if $\alpha_1 x\neq 0$ for any $\alpha_1\in\mathfrak{o}_k$ such that $\alpha_1\notin\alpha\mathfrak{o}_k$. Plainly, $\mathrm{Ker}(\alpha)=\alpha^{-1}\Omega/\Omega$ and this is a finite group of order $N\alpha$, where $N\alpha$ means the absolute norm of α. Moreover, if $x_\alpha\in\mathrm{Ker}(\alpha)$ is a fixed primitive α-division point, then the map

$$\mathfrak{o}_k \;\to\; \boldsymbol{C}/\Omega \quad\text{by}\quad r \;\mapsto\; r x_\alpha$$

induces an isomorphism of $\mathfrak{o}_k$-module from $\mathfrak{o}_k/\alpha\mathfrak{o}_k$ onto $\mathrm{Ker}(\alpha)$, and $\mathrm{Ker}(\alpha)$ may be written as $\{\,r x_\alpha \mid r \bmod \alpha\mathfrak{o}_k,\ r\in\mathfrak{o}_k\,\}$. Sometimes, for convenience sake, we use the notation $\mathrm{Ker}(\alpha)$, identifing it with a complete set of representatives of $\alpha^{-1}\Omega/\Omega$.

Using the formula (D8), we can restate our product formula for f_Ω (Theorem 1-3 in [6]) as follows.

Theorem 1-2 (Revised product formula). *For any α in $J^*(2)$,*

$$f_\Omega(\alpha z)\frac{D_\Omega^2\big(\,\alpha z\,;\,\alpha\dfrac{\omega_1+\omega_2}{2}\,\big)}{D_\Omega^2\big(\,\alpha z\,;\,\dfrac{\omega_1+\omega_2}{2}\,\big)} = \xi_\Omega(\alpha)\prod_{x\in\mathrm{Ker}(\alpha)} f_\Omega(z+x),$$

where $\xi_\Omega(\alpha)$ *is given by*

$$\xi_\Omega(\alpha) = \alpha \prod_{\substack{x\in \mathrm{Ker}(\alpha)\\ x\neq 0}} \big(f_\Omega(x)\big)^{-1} = \alpha \prod_{\substack{x\in \mathrm{Ker}(\alpha)\\ x\neq 0}} f_\Omega\big(x + \frac{\omega_1}{2}\big).$$

Here we remark that if $\alpha \equiv 1 \pmod{2\mathfrak{o}_k}$, D_Ω-factor in Theorem 1-2 can be deleted and we have $\xi_\Omega^2(\alpha) = 1$ as in Bayad [1]. In the rest of this section, we consider the value $\xi_\Omega(\alpha)$ more precisely, reviewing our arguments in Sec. 1 of [6].

Now the multiplicative group $(\mathfrak{o}_k/2\mathfrak{o}_k)^\times$ has 3 possibilities:

(a) $(\mathfrak{o}_k/2\mathfrak{o}_k)^\times \cong \{1\}$, when 2 splits in k.

(b) $(\mathfrak{o}_k/2\mathfrak{o}_k)^\times$ is a cyclic group of order 2, when 2 ramifies in k.

(c) $(\mathfrak{o}_k/2\mathfrak{o}_k)^\times$ is a cyclic group of order 3, when 2 remains prime in k.

In case (a), since $\alpha \equiv 1 \pmod{2\mathfrak{o}_k}$ for any $\alpha \in J^*(2)$, $\xi_\Omega^2(\alpha) = 1$ as in [1].

In case (b), by a suitable choice of basis $\{\omega_1, \omega_2\}$ of Ω, we may assume that $\omega_1/2$ and $\omega_2/2$ represent two distinct primitive 2-division points of Ω and $(\omega_1+\omega_2)/2$ other non-zero 2-division point. Under this assumption, we showed that $\xi_\Omega^2(\alpha) = -1$ for any $\alpha \in J^*(2)$ such that $\alpha \not\equiv 1 \pmod{2\mathfrak{o}_k}$.

In case (c), we showed that for any $\alpha \in J^*(2)$ such that $\alpha \not\equiv 1 \pmod{2\mathfrak{o}_k}$

$$(\alpha^2+\alpha+1)\,\frac{\omega_1+\omega_2}{2} \equiv 0 \pmod{\Omega}$$

and

$$\xi_\Omega^2(\alpha) = -\frac{\wp_\Omega\big(\alpha^2\frac{\omega_1+\omega_2}{2}\big) - \wp_\Omega\big(\alpha\frac{\omega_1+\omega_2}{2}\big)}{\wp_\Omega\big(\alpha^2\frac{\omega_1+\omega_2}{2}\big) - \wp_\Omega\big(\frac{\omega_1+\omega_2}{2}\big)}.$$

$$= -\frac{D_\Omega^2\big(\alpha^2\frac{\omega_1+\omega_2}{2}\,;\,\alpha\,\frac{\omega_1+\omega_2}{2}\big)}{D_\Omega^2\big(\alpha^2\frac{\omega_1+\omega_2}{2}\,;\,\frac{\omega_1+\omega_2}{2}\big)}.$$

Herein $\xi_\Omega^2(\alpha)$ gives a unit in $k(2)$, the ray class field over k with conductor $2\mathfrak{o}_k$. For simplicity, we let $\tau = (\omega_1+\omega_2)/2$ and $(\alpha^2+\alpha+1)\,\tau = u$ with some $u \in \Omega$. Then, using the formulas (K1) and (1.1), $\xi_\Omega^2(\alpha)$ can be further reformed as follows.

$$\begin{aligned}
\xi_\Omega^2(\alpha) &= -\frac{D_\Omega^2(\alpha^2\tau\,;\,\alpha\tau)}{D_\Omega^2(\alpha^2\tau\,;\,\tau)}\\
&= -e\big(E_\Omega(\alpha^2\tau,\alpha\tau) - E_\Omega(\alpha^2\tau,\tau)\big)\,\frac{\mathcal{K}_\Omega^2((\alpha^2+\alpha)\tau)}{\mathcal{K}_\Omega^2(\alpha^2\tau)\,\mathcal{K}_\Omega^2(\alpha\tau)}\,\frac{\mathcal{K}_\Omega^2(\alpha^2\tau)\,\mathcal{K}_\Omega^2(\tau)}{\mathcal{K}_\Omega^2((\alpha^2+1)\tau)}\\
&= -e\big(E_\Omega(\alpha^2\tau,(\alpha-1)\tau)\big)\,\frac{\mathcal{K}_\Omega^2(-\tau+u)\,\mathcal{K}_\Omega^2(\tau)}{\mathcal{K}_\Omega^2(\alpha\tau)\,\mathcal{K}_\Omega^2(-\alpha\tau+u)}
\end{aligned}$$

$$= -e\big(2\, E_\Omega(\alpha\tau,\tau)\big)\ \frac{\mathcal{K}_\Omega^4(\tau)}{\mathcal{K}_\Omega^4(\alpha\tau)}.$$

Hereby

$$\begin{aligned} 2\, E_\Omega(\alpha\tau,\tau) &= E_\Omega(\alpha\, \frac{\omega_1+\omega_2}{2},\, \omega_1+\omega_2) \\ &\equiv E_\Omega\big(\frac{\omega_1}{2},\, \omega_1+\omega_2\big) \text{ or } \ E_\Omega\big(\frac{\omega_2}{2},\, \omega_1+\omega_2\big) \pmod{\boldsymbol{Z}} \\ &\equiv \frac{1}{2} \pmod{\boldsymbol{Z}}. \end{aligned}$$

and hence $e\big(\,2E_\Omega(\alpha\tau,\tau)\,\big) = -1$. Then we have

$$\xi_\Omega^2(\alpha) = \frac{\mathcal{K}_\Omega^4\big(\frac{\omega_1+\omega_2}{2}\big)}{\mathcal{K}_\Omega^4\big(\alpha\, \frac{\omega_1+\omega_2}{2}\big)}. \tag{1.3}$$

2. Characters ε_Ω and $\tilde{\varepsilon}_\Omega$.

As in Section 1, let Ω be an $\mathfrak{o}_k$-ideal and let $\{\omega_1, \omega_2\}$ a fixed basis of Ω . Then, using the formulas (1.2), (K2) and (K4), we have

$$\prod_{\substack{x\in \mathrm{Ker}(\alpha)\\ x\neq 0}} f_\Omega(x)$$

$$= e\big(\frac{1}{8}(N\alpha-1)\, E_\Omega(\omega_2,\omega_1)\big) \prod_{\substack{x\in \mathrm{Ker}(\alpha)\\ x\neq 0}} \frac{\mathcal{K}_\Omega(x)\mathcal{K}_\Omega\big(x+\frac{\omega_1+\omega_2}{2}\big)}{\mathcal{K}_\Omega\big(x+\frac{\omega_1}{2}\big)\, \mathcal{K}_\Omega\big(x+\frac{\omega_2}{2}\big)}$$

$$= e\big(\frac{1}{8}(N\alpha-1)\, E_\Omega(\omega_2,\omega_1)\big)\ \times$$
$$\prod_{\substack{x\in \mathrm{Ker}(\alpha)\\ x\neq 0}} \frac{\mathcal{K}_\Omega\big(x+\frac{\omega_1+\omega_2}{2}\big)}{\mathcal{K}_\Omega(x)}\ \frac{\mathcal{K}_\Omega(x)}{\mathcal{K}_\Omega\big(x+\frac{\omega_1}{2}\big)}\ \frac{\mathcal{K}_\Omega(x)}{\mathcal{K}_\Omega\big(x+\frac{\omega_2}{2}\big)}$$

$$= e\big(\frac{1}{8}(N\alpha-1)\, E_\Omega(\omega_2,\omega_1)\big)\ \frac{\mathcal{K}_{\alpha^{-1}\Omega}\big(\frac{\omega_1+\omega_2}{2}\big)}{\mathcal{K}_\Omega\big(\frac{\omega_1+\omega_2}{2}\big)}\ \frac{\mathcal{K}_\Omega\big(\frac{\omega_1}{2}\big)}{\mathcal{K}_{\alpha^{-1}\Omega}\big(\frac{\omega_1}{2}\big)}\ \frac{\mathcal{K}_\Omega\big(\frac{\omega_2}{2}\big)}{\mathcal{K}_{\alpha^{-1}\Omega}\big(\frac{\omega_2}{2}\big)}$$

$$= e\big(\frac{1}{8}(N\alpha-1)\, E_\Omega(\omega_2,\omega_1)\big)\ \alpha\ \frac{\mathcal{K}_\Omega\big(\alpha\, \frac{\omega_1+\omega_2}{2}\big)}{\mathcal{K}_\Omega\big(\frac{\omega_1+\omega_2}{2}\big)}\ \frac{\mathcal{K}_\Omega\big(\frac{\omega_1}{2}\big)}{\mathcal{K}_\Omega\big(\alpha\, \frac{\omega_1}{2}\big)}\ \frac{\mathcal{K}_\Omega\big(\frac{\omega_2}{2}\big)}{\mathcal{K}_\Omega\big(\alpha\, \frac{\omega_2}{2}\big)}.$$

Hence

$$\begin{aligned}\xi_\Omega(\alpha) &= \alpha\Big(\prod_{\substack{x\in\mathrm{Ker}(\alpha)\\ x\neq 0}} f_\Omega(x)\Big)^{-1}\\ &= e\big(\frac{1}{8}(N\alpha-1)\,E_\Omega(\omega_1,\omega_2)\big)\ \frac{\mathcal{K}_\Omega(\frac{\omega_1+\omega_2}{2})}{\mathcal{K}_\Omega(\alpha\frac{\omega_1+\omega_2}{2})}\,\frac{\mathcal{K}_\Omega(\alpha\frac{\omega_1}{2})}{\mathcal{K}_\Omega(\frac{\omega_1}{2})}\,\frac{\mathcal{K}_\Omega(\alpha\frac{\omega_2}{2})}{\mathcal{K}_\Omega(\frac{\omega_2}{2})}.\end{aligned}$$

Here we define ε_Ω by

$$\varepsilon_\Omega(\alpha) := e\big(\frac{1}{8}(N\alpha-1)\,E_\Omega(\omega_1,\omega_2)\,\big)\prod_\rho \frac{\mathcal{K}_\Omega(\alpha\rho)}{\mathcal{K}_\Omega(\rho)} \tag{2.1}$$

for $\alpha\in J^*(2)$, where ρ runs over the set $\{\,\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}\,\}$. Then we have

$$\xi_\Omega(\alpha) = \varepsilon_\Omega(\alpha)\,\frac{\mathcal{K}_\Omega^2(\,\frac{\omega_1+\omega_2}{2}\,)}{\mathcal{K}_\Omega^2(\,\alpha\,\frac{\omega_1+\omega_2}{2}\,)}. \tag{2.2}$$

From the definition, it is easy to see that $\varepsilon_\Omega^4(\alpha)=1$. Of course, the definition of ε_Ω depends on the basis $\{\omega_1,\omega_2\}$ of Ω. Indeed, by a short calculation, we have the following

Lemma 2-1. *Any of three substitutions $(\omega_1,\omega_2)\to(\omega_2,\omega_1)$, $(\omega_1,\omega_2)\to(\omega_2,-\omega_1)$ and $(\omega_1,\omega_2)\to(\omega_1,\omega_1+\omega_2)$ do multiply $\varepsilon_\Omega(\alpha)$ by the quantity*

$$\chi_4\circ N(\alpha) = \chi_4(N\alpha) := (-1)^{\frac{1}{2}(N\alpha-1)}.$$

Remark. $\chi_4\circ N$ is a quadratic character of $(\mathfrak{o}_k/4\mathfrak{o}_k)^\times$. In particular, in the case where $N\alpha\equiv 1 \pmod 4$ for any $\alpha\in J^*(2)$, the definition of ε_Ω does not depend on the choice of basis $\{\omega_1,\omega_2\}$ of Ω.

In case (a) where $(\mathfrak{o}_k/2\mathfrak{o}_k)^\times\cong\{1\}$, since $\alpha\equiv 1 \pmod{2O}$ for any $\alpha\in J^*(2)$, we have $\varepsilon_\Omega^2(\alpha)=\xi_\Omega^2(\alpha)=1$.

In case (b) where $(\mathfrak{o}_k/2\mathfrak{o}_k)^\times$ is a group of order 2, we first choose a basis $\{\omega_1,\omega_2\}$ of Ω so that $\omega_1/2$ and $\omega_2/2$ represent two distinct primitive 2-division points of Ω. Then for any $\alpha\in J^*(2)$, we have

$$\alpha\,\frac{\omega_1+\omega_2}{2}\equiv\frac{\omega_1+\omega_2}{2} \pmod{\Omega}$$

and $\varepsilon_\Omega^2(\alpha)=\xi_\Omega^2(\alpha)$ by the formulas (2.2) and (K1). Especially in this case, if $\alpha\not\equiv 1 \pmod{2\mathfrak{o}_k}$, then $\varepsilon_\Omega^2(\alpha)=-1$ (see Sec.1). Moreover, by Lemma 2-1, the same assertion holds without any restriction on the choice of basis of Ω.

In case (c) where $(\mathfrak{o}_k/2\mathfrak{o}_k)^\times$ is a group of order 3, from the equations (1.3) and (2.2), we see that $\varepsilon_\Omega^2(\alpha)=1$ for any $\alpha\in J^*(2)$.

Consequently we see that in both cases (a) and (c) ε_Ω takes value in $\{\pm 1\}$, and in the remaining case (b) it takes value in $\{\,\pm 1,\pm\sqrt{-1}\,\}$. Moreover, we have the following

Proposition 2-2. *For any $\alpha\in J^*(2)$, $\varepsilon_\Omega(\alpha)$ is determined depending only on the class of α modulo $4\mathfrak{o}_k$.*

Proof. Let's assume that $\alpha_1\equiv\alpha \pmod{4\mathfrak{o}_k}$, i.e. $\alpha_1=\alpha+4u$ with some $u\in\mathfrak{o}_k$. Then, on the one hand, since

$$N\alpha_1 = N\alpha + 4Tr(\overline{\alpha}u) + 16Nu\,,$$

we have

$$\begin{aligned}&e\big(\,\frac{1}{8}(N\alpha_1-1)E_\Omega(\omega_1,\omega_2)\,\big)\\ &= e\big(\,\frac{1}{8}(N\alpha-1)E_\Omega(\omega_1,\omega_2)\,\big)\cdot e\big(\,\frac{1}{2}Tr(\overline{\alpha}u)\,E_\Omega(\omega_1,\omega_2)\,\big),\end{aligned}$$

where Tr means the trace map as usual. On the other hand, by the formula (K1)

$$\prod_\rho \mathcal{K}_\Omega(\alpha_1\rho) = \prod_\rho \mathcal{K}_\Omega(\alpha\rho+4u\rho) = e(M)\ \prod_\rho \mathcal{K}_\Omega(\alpha\rho)\,,$$

where ρ runs over the set $\{\,\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_1+\omega_2}{2}\,\}$ and M is given by

$$M = 2\,\big(\sum_\rho N\rho\,\big)\cdot E_\Omega(u,\alpha).$$

Moreover a short calculation shows that

$$\begin{aligned} M &\equiv \frac{1}{2}(\overline{\omega_1}\omega_2+\omega_1\overline{\omega_2})\,E_\Omega(u,\alpha) \pmod{\boldsymbol{Z}}\\ &\equiv \frac{1}{2}(\overline{u}\alpha+u\overline{\alpha})\,E_\Omega(\omega_1,\omega_2) \pmod{\boldsymbol{Z}}\\ &= \frac{1}{2}Tr(\overline{u}\alpha)\,E_\Omega(\omega_1,\omega_2)\,.\end{aligned}$$

Hence we have $\varepsilon_\Omega(\alpha_1)=\varepsilon_\Omega(\alpha)$.

Remark. In the same way as in the proof of Proposition 2-2, we can see that $\varepsilon_\Omega^2(\alpha)$ is determined depending only on the class of α modulo $2\mathfrak{o}_k$.

Proposition 2-2 suggests an expectation that ε_Ω could be a character of $(\mathfrak{o}_k/4\mathfrak{o}_k)^\times$. However it is not true in general. Namely, in the next section, we shall prove the following

Theorem 2-3. $\varepsilon_\Omega(\alpha\beta) = \varepsilon_\Omega(\alpha)^{N\beta}\varepsilon_\Omega(\beta) = \varepsilon_\Omega(\alpha)\varepsilon_\Omega(\beta)^{N\alpha}$ *for any* $\alpha\,,\,\beta \in J^*(2)$.

Theorem 2-3 illustrates an action of $\mathrm{Gal}(\,k^{ab}/\mathcal{H}\,)$ on $\varepsilon_\Omega(\alpha)$, where $\mathcal{H}$ is the Hilbert class field over k. Namely, we let $\sigma(\beta) := (\,(\beta)\,,\,k^{ab}/\mathcal{H}\,)$, the Artin automorphism belonging to the principal ideal $(\beta) = \beta\mathfrak{o}_k$. Then,

$$\varepsilon_\Omega(\alpha)^{\sigma(\beta)} = \varepsilon_\Omega(\alpha)^{N\beta} = \frac{\varepsilon_\Omega(\alpha\beta)}{\varepsilon_\Omega(\beta)}. \tag{2.3}$$

At any rate, as a consequence of Theorem 2-3, we see that in both cases (a) and (c) ε_Ω defines a character on $(\mathfrak{o}_k/4\mathfrak{o}_k)^\times$ of order 2. Also in case (b), if $N\alpha \equiv 1 \pmod 4$ for any $\alpha \in J^*(2)$, then ε_Ω defines a character on $(\mathfrak{o}_k/4\mathfrak{o}_k)^\times$ of order 4.

Now let $-d_k$ be the discriminant of k. Then we are in case (b) if and only if $4 \mid d_k$. Moreover, if $d_k = 4d_0$ with $d_0 \equiv 1 \pmod 4$, it always holds that $N\alpha \equiv 1 \pmod 4$ for any $\alpha \in J^*(2)$. However in the case where $d_k = 4d_0$ with $d_0 \equiv 2 \pmod 4$, we see that $N\alpha \equiv -1 \pmod 4$ for any $\alpha \in J^*(2)$ such that $\alpha \not\equiv 1 \pmod{2\mathfrak{o}_k}$. Hence in this case, ε_Ω cannot be a character on $(\mathfrak{o}_k/4\mathfrak{o}_k)^\times$. Indeed, by Theorem 2-3, we have

$$\frac{\varepsilon_\Omega(\alpha\beta)}{\varepsilon_\Omega(\alpha)\varepsilon_\Omega(\beta)} = \varepsilon_\Omega(\alpha)^{N\beta-1} = -1$$

for any $\alpha\,,\,\beta \in J^*(2)$ such that $\alpha \not\equiv 1\,,\,\beta \not\equiv 1 \pmod{2\mathfrak{o}_k}$. In the case where $d_k = 4d_0$ with $d_0 \equiv 2 \pmod 4$, in place of ε_Ω, we consider $\tilde{\varepsilon}_\Omega$ defined by

$$\tilde{\varepsilon}_\Omega(\alpha) := e\Big(\,\frac{1}{8}(N\alpha - 1)\,\Big)\,\varepsilon_\Omega(\alpha).$$

Then $\tilde{\varepsilon}_\Omega$ also satisfies the cocycle property, i.e.

$$\tilde{\varepsilon}_\Omega(\alpha\beta) = \tilde{\varepsilon}_\Omega(\alpha)^{N\beta}\tilde{\varepsilon}_\Omega(\beta) = \tilde{\varepsilon}_\Omega(\alpha)\tilde{\varepsilon}_\Omega(\beta)^{N\alpha}$$

for any $\alpha, \beta \in J^*(2)$. It is also true that the value $\tilde{\varepsilon}_\Omega(\alpha)$ depends only on the class of α modulo $4\mathfrak{o}_k$ and further $\tilde{\varepsilon}_\Omega^2(\alpha) = 1$ for any $\alpha \in J^*(2)$. Indeed, by Remark to Proposition 2-2, $\varepsilon_\Omega^2(\alpha) = -1$ if and only if $\alpha \not\equiv 1 \pmod{2\mathfrak{o}_k}$ and equivalently $N\alpha \not\equiv 1 \pmod 4$. Hence $\tilde{\varepsilon}_\Omega$ defines a character on $(\mathfrak{o}_k/4\mathfrak{o}_k)^\times$ of order 2. We will use this modified character $\tilde{\varepsilon}_\Omega$ in Section 4.

3. Proof of Theorem 2-3.

Let all notations be the same as those in Section 2. For a complete proof of Theorem 2-3, it suffices to prove a half part of equalites, i.e.

$$\varepsilon_\Omega(\alpha\beta) = \varepsilon_\Omega(\alpha)\varepsilon_\Omega(\beta)^{N\alpha},$$

that is equivalent to the equality

$$\xi_\Omega(\alpha\beta) = \xi_\Omega(\alpha)\,\xi_\Omega(\beta)^{N\alpha}\left(\frac{\mathcal{K}_\Omega^2(\beta\frac{\omega_1+\omega_2}{2})}{\mathcal{K}_\Omega^2(\frac{\omega_1+\omega_2}{2})}\right)^{N\alpha}\frac{\mathcal{K}_\Omega^2(\alpha\frac{\omega_1+\omega_2}{2})}{\mathcal{K}_\Omega^2(\alpha\beta\frac{\omega_1+\omega_2}{2})}.$$

For this purpose we can apply the product formula of the Bayad function f_Ω in Theorem 1-2. For simplicity, we let $\tau = (\omega_1+\omega_2)/2$ again. Then on the one hand, we have

$$\begin{aligned} f_\Omega(\alpha\beta z)\frac{D_\Omega^2(\alpha\beta z\,;\,\alpha\beta\tau)}{D_\Omega^2(\alpha\beta z\,;\,\tau)} &= \xi_\Omega(\alpha\beta)\prod_{x\in\mathrm{Ker}(\alpha\beta)} f_\Omega(z+x) \\ &= \xi_\Omega(\alpha\beta)\prod_{\substack{\tilde r \bmod \alpha\beta\mathfrak{o}_k \\ \tilde r\in\mathfrak{o}_k}} f_\Omega(z+\tilde r\,x_{\alpha\beta}), \end{aligned}$$

where $x_{\alpha\beta}$ is a fixed primitive $\alpha\beta$-division point of $\boldsymbol{C}/\Omega$. On the other hand,

$$\begin{aligned} f_\Omega(\alpha\beta z)\frac{D_\Omega^2(\alpha\beta z\,;\,\alpha\beta\tau)}{D_\Omega^2(\alpha\beta z\,;\,\tau)} &= \frac{D_\Omega^2(\alpha\beta z\,;\,\alpha\beta\tau)}{D_\Omega^2(\alpha\beta z\,;\,\alpha\tau)} \times f_\Omega(\,\alpha(\beta z)\,)\frac{D_\Omega^2(\,\alpha(\beta z)\,;\,\alpha\tau)}{D_\Omega^2(\,\alpha(\beta z)\,;\,\tau)} \\ &= \frac{D_\Omega^2(\alpha\beta z\,;\,\alpha\beta\tau)}{D_\Omega^2(\alpha\beta z\,;\,\alpha\tau)} \times \xi_\Omega(\alpha)\prod_{\substack{r_1 \bmod \alpha\mathfrak{o}_k \\ r_1\in\mathfrak{o}_k}} f_\Omega(\beta z + r_1\,x_\alpha), \end{aligned}$$

where $x_\alpha := \beta\,x_{\alpha\beta}$ and this gives a primitive α-division point of $\boldsymbol{C}/\Omega$. Moreover, in the above last equality,

$$\begin{aligned} &\prod_{\substack{r_1 \bmod \alpha\mathfrak{o}_k \\ r_1\in\mathfrak{o}_k}} f_\Omega(\beta z + r_1\,x_\alpha) \\ &= \prod_{\substack{r_1 \bmod \alpha\mathfrak{o}_k \\ r_1\in\mathfrak{o}_k}} f_\Omega(\,\beta(z+r_1\,x_{\alpha\beta})\,)\frac{D_\Omega^2(\,\beta(z+r_1x_{\alpha\beta})\,;\,\beta\tau)}{D_\Omega^2(\,\beta(z+r_1x_{\alpha\beta})\,;\,\tau\,)} \\ &\quad\times \prod_{\substack{r_1 \bmod \alpha\mathfrak{o}_k \\ r_1\in\mathfrak{o}_k}} \frac{D_\Omega^2(\,\beta z+r_1x_\alpha\,;\,\tau)}{D_\Omega^2(\,\beta z+r_1x_\alpha\,;\,\beta\tau\,)} \\ &= \xi_\Omega(\beta)^{N\alpha}\prod_{\substack{r_1 \bmod \alpha\mathfrak{o}_k \\ r_1\in\mathfrak{o}_k}} \frac{D_\Omega^2(\,\beta z+r_1x_\alpha\,;\,\tau)}{D_\Omega^2(\,\beta z+r_1x_\alpha\,;\,\beta\tau\,)} \times \prod_{\substack{r_1 \bmod \alpha\mathfrak{o}_k \\ r_2 \bmod \beta\mathfrak{o}_k}} f_\Omega(\,z+r_1x_{\alpha\beta}+r_2x_\beta\,) \end{aligned}$$

$$= \xi_\Omega(\beta)^{N\alpha} \prod_{\substack{r_1 \bmod \alpha\mathfrak{o}_k \\ r_1 \in \mathfrak{o}_k}} \frac{D_\Omega^2(\,\beta z + r_1 x_\alpha\,;\,\tau\,)}{D_\Omega^2(\,\beta z + r_1 x_\alpha\,;\,\beta\tau\,)} \;\times \prod_{\tilde{r} \bmod \alpha\beta\mathfrak{o}_k} f_\Omega(\,z + \tilde{r} x_{\alpha\beta}\,).$$

Herein $x_\beta := \alpha\, x_{\alpha\beta}$ and this gives a primitive β-division point of $\boldsymbol{C}/\Omega$. Note that $\{\,r_1 + \alpha r_2 \;\mid\; r_1 \bmod \alpha\mathfrak{o}_k,\; r_2 \bmod \beta\mathfrak{o}_k\,\}$ constitutes a complete system of representatives of $\mathfrak{o}_k/\alpha\beta\mathfrak{o}_k$. Hence we obtain the following equality

$$\xi_\Omega(\alpha\beta) = \xi_\Omega(\alpha)\xi_\Omega(\beta)^{N\alpha} \;\times\; F_\Omega(\,z\,;\,\alpha,\beta\,)$$

where

$$F_\Omega(\,z\,;\,\alpha,\beta\,) = \frac{D_\Omega^2(\alpha\beta z\,;\,\alpha\beta\tau)}{D_\Omega^2(\alpha\beta z\,;\,\alpha\tau)} \;\times \prod_{\substack{r_1 \bmod \alpha\mathfrak{o}_k \\ r_1 \in \mathfrak{o}_k}} \frac{D_\Omega^2(\,\beta z + r_1 x_\alpha\,;\,\tau\,)}{D_\Omega^2(\,\beta z + r_1 x_\alpha\,;\,\beta\tau\,)}.$$

Moreover, by the formula (D6), we have

$$\begin{aligned} F_\Omega(\,z\,;\,\alpha,\beta\,) \;&= \frac{D_\Omega^2(\alpha\beta z\,;\,\alpha\beta\tau)}{D_\Omega^2(\alpha\beta z\,;\,\alpha\tau)} \cdot \frac{D_{\alpha^{-1}\Omega}^2(\,\beta z\,;\,\tau)}{D_{\alpha^{-1}\Omega}^2(\,\beta z\,;\,\beta\tau)} \cdot \frac{\mathcal{K}_\Omega^2(\beta\tau)^{N\alpha}}{\mathcal{K}_{\alpha^{-1}\Omega}^2(\beta\tau)} \cdot \frac{\mathcal{K}_{\alpha^{-1}\Omega}^2(\tau)}{\mathcal{K}_\Omega^2(\tau)^{N\alpha}} \\ &\quad \times e\Big(\,E_\Omega\big(\sum_{r_1 \bmod \alpha\mathfrak{o}_k} r_1 x_\alpha\;,\;2(1-\beta)\tau\,\big)\Big) \\ &= \frac{\mathcal{K}_\Omega^2(\beta\tau)^{N\alpha}}{\mathcal{K}_\Omega^2(\alpha\beta\tau)} \cdot \frac{\mathcal{K}_\Omega^2(\alpha\tau)}{\mathcal{K}_\Omega^2(\tau)^{N\alpha}} \\ &= \left(\frac{\mathcal{K}_\Omega^2\big(\,\beta\dfrac{\omega_1+\omega_2}{2}\,\big)}{\mathcal{K}_\Omega^2\big(\,\dfrac{\omega_1+\omega_2}{2}\,\big)}\right)^{N\alpha} \frac{\mathcal{K}_\Omega^2\big(\,\alpha\dfrac{\omega_1+\omega_2}{2}\,\big)}{\mathcal{K}_\Omega^2\big(\,\alpha\beta\dfrac{\omega_1+\omega_2}{2}\,\big)}, \end{aligned}$$

and this proves Theorem 2-3.

4. Quadratic reciprocity law.

Let all notations be the same as those in the preceding sections. We fix a basis $\{\omega_1, \omega_2\}$ of an $\mathfrak{o}_k$-ideal Ω. For any α, β in $J^*(2)$ such that $(\alpha, \beta) = 1$, we consider the quadratic symbol $\big(\dfrac{\alpha}{\beta}\big)_2$ given by

$$\left(\frac{\alpha}{\beta}\right)_2 = \prod_{x \in S_\beta} \varepsilon(\alpha, x),$$

where S_β means a subset of $\mathrm{Ker}(\beta) = \mathrm{Ker}_\Omega(\beta)$ such that $\mathrm{Ker}(\beta) = \{0,\; S_\beta, -S_\beta\}$, and $\varepsilon(\alpha, x) \in \{\pm 1\}$ is so determined that $\alpha x = \varepsilon(\alpha, x)\gamma(x)$ with unique $\gamma(x)$ in S_β. Note that this definition of $(\dfrac{\alpha}{\beta})_2$ does not depend on the choice of S_β (cf. [9]).

In [6], we gave a revised version of Bayad's reciprocity formula on $(\dfrac{\alpha}{\beta})_2$ (Theorem 2-1 in [6]). In this opportunity we further reformulte our formula in the following form.

Theorem 4-1. *For $\alpha, \beta \in J^*(2)$ such that $(\alpha, \beta) = 1$,*

$$\left(\frac{\alpha}{\beta}\right)_2 \left(\frac{\beta}{\alpha}\right)_2 = (-1)^{\frac{1}{4}(N\alpha-1)(N\beta-1)} \times \frac{\varepsilon_\Omega(\alpha)^{\frac{1}{2}(N\beta-1)}}{\varepsilon_\Omega(\beta)^{\frac{1}{2}(N\alpha-1)}}.$$

Proof. Since f_Ω is an odd function, $f_\Omega(\alpha x) = \varepsilon(\alpha, x) f_\Omega(\gamma(x))$ for any $x \in S_\beta$. Then we have

$$\left(\frac{\alpha}{\beta}\right)_2 = \prod_{x \in S_\beta} \varepsilon(\alpha, x) = \prod_{x \in S_\beta} \frac{f_\Omega(\alpha x)}{f_\Omega(\gamma(x))} = \prod_{x \in S_\beta} \frac{f_\Omega(\alpha x)}{f_\Omega(x)}.$$

Moreover, using the product formula in Theorem 1-2, we have

$$\begin{aligned}
\left(\frac{\alpha}{\beta}\right)_2 &= \prod_{x \in S_\beta} \left(\frac{D_\Omega^2(\,\alpha x\,;\, \dfrac{\omega_1+\omega_2}{2}\,)}{D_\Omega^2(\,\alpha x\,;\, \alpha\dfrac{\omega_1+\omega_2}{2})}\, \xi_\Omega(\alpha) \prod_{\substack{x' \in \mathrm{Ker}(\alpha) \\ x' \neq 0}} f_\Omega(x + x') \right) \\
&= \xi_\Omega(\alpha)^{\frac{N\beta-1}{2}} A_\beta^{(\alpha)} \prod_{x \in S_\beta} \prod_{x' \in S_\alpha} f_\Omega(x + x') f_\Omega(x - x'),
\end{aligned}$$

where

$$A_\beta^{(\alpha)} = \prod_{x \in S_\beta} \frac{D_\Omega^2(\,\alpha x\,;\,\tau\,)}{D_\Omega^2(\,\alpha x\,;\,\alpha\tau\,)} \qquad \text{with} \quad \tau = \frac{\omega_1+\omega_2}{2}.$$

By the formula (D8), we have

$$\begin{aligned}
A_\beta^{(\alpha)} &= \prod_{x \in S_\beta} \frac{\wp_\Omega(\alpha x) - \wp_\Omega(\tau)}{\wp_\Omega(\alpha x) - \wp_\Omega(\alpha\tau)} && (\text{original form in } [6]) \\
&= \prod_{x \in S_\beta} \frac{\wp_\Omega(\tau) - \wp_\Omega(x)}{\wp_\Omega(\alpha\tau) - \wp_\Omega(x)} && (\,\wp_\Omega \text{ is even and } \Omega \text{ ellitic}\,) \\
&= \prod_{x \in S_\beta} \frac{D_\Omega(\tau; x)\, D_\Omega(\tau; -x)}{D_\Omega(\alpha\tau; x)\, D_\Omega(\alpha\tau; -x)} \\
&= \prod_{\substack{x \in \mathrm{Ker}(\beta) \\ x \neq 0}} \frac{D_\Omega(\tau; x)}{D_\Omega(\alpha\tau; x)},
\end{aligned}$$

and then using the formulas (D4) and (D7) (with $\Lambda = \beta^{-1}\Omega$)

$$A_\beta^{(\alpha)} = \frac{\mathcal{K}_\Omega(\alpha\tau)^{N\beta}}{\mathcal{K}_{\beta^{-1}\Omega}(\alpha\tau)} \cdot \frac{\mathcal{K}_{\beta^{-1}\Omega}(\tau)}{\mathcal{K}_\Omega(\tau)^{N\beta}} = \left\{\frac{\mathcal{K}_\Omega(\alpha\tau)}{\mathcal{K}_\Omega(\tau)}\right\}^{N\beta} \cdot \frac{\mathcal{K}_\Omega(\beta\tau)}{\mathcal{K}_\Omega(\alpha\beta\tau)}$$

$$= H(\,\alpha,\beta\,;\,\tau\,)\ \left\{\frac{\mathcal{K}_\Omega(\alpha\tau)}{\mathcal{K}_\Omega(\tau)}\right\}^{N\beta-1}.$$

Herein

$$H(\,\alpha,\beta\,;\,\tau\,) := \frac{\mathcal{K}_\Omega(\alpha\tau)\,\mathcal{K}_\Omega(\beta\tau)}{\mathcal{K}_\Omega(\tau)\,\mathcal{K}_\Omega(\alpha\beta\tau)}.$$

Note that $H(\,\alpha,\beta\,;\,\tau\,) = H(\,\beta,\alpha\,;\,\tau\,)$. Moreover, since

$$\xi_\Omega(\alpha) = \varepsilon_\Omega(\alpha)\frac{\mathcal{K}_\Omega^2(\tau)}{\mathcal{K}_\Omega^2(\alpha\tau)},$$

we have

$$\xi_\Omega(\alpha)^{\frac{1}{2}(N\beta-1)}\,A_\beta^{(\alpha)} = H(\,\alpha,\beta\,;\,\tau\,)\,\varepsilon_\Omega(\alpha)^{\frac{1}{2}(N\beta-1)},$$

and hence

$$\left(\frac{\alpha}{\beta}\right)_2 = H(\,\alpha,\beta\,;\,\tau\,)\,\varepsilon_\Omega(\alpha)^{\frac{1}{2}(N\beta-1)} \times \prod_{x\in S_\beta}\prod_{x'\in S_\alpha} f_\Omega(x+x')f_\Omega(x-x').$$

Symmetrically we have

$$\left(\frac{\beta}{\alpha}\right)_2 = H(\,\beta\,,\,\alpha\,;\tau\,)\,\varepsilon_\Omega(\beta)^{\frac{1}{2}(N\alpha-1)} \prod_{x'\in S_\alpha}\prod_{x\in S_\beta} f_\Omega(x'+x)f_\Omega(x'-x)$$

$$= H(\,\alpha\,,\,\beta\,;\tau\,)\,\varepsilon_\Omega(\beta)^{\frac{1}{2}(N\alpha-1)}\ (-1)^{\frac{1}{4}(N\alpha-1)(N\beta-1)}$$

$$\times \prod_{x\in S_\beta}\prod_{x'\in S_\alpha} f_\Omega(x+x')f_\Omega(x-x'),$$

and hence

$$\left(\frac{\alpha}{\beta}\right)_2\left(\frac{\beta}{\alpha}\right)_2 = \left(\frac{\alpha}{\beta}\right)_2\left(\frac{\beta}{\alpha}\right)_2^{-1} = (-1)^{\frac{1}{4}(N\alpha-1)(N\beta-1)} \times \frac{\varepsilon_\Omega(\alpha)^{\frac{1}{2}(N\beta-1)}}{\varepsilon_\Omega(\beta)^{\frac{1}{2}(N\alpha-1)}}.$$

Thus we have furnished the proof of Theorem 4-1.

Remark. As is explained in Section 2, ε_Ω is a character on $(\mathfrak{o}_k/4\mathfrak{o}_k)^\times$ except for the case where $d_k = 4d_0$ and $d_0 \equiv 2 \pmod 4$. In the exceptional case, we may replace ε_Ω in Theorem 4-1 by a character $\tilde{\varepsilon}_\Omega$ on $(\mathfrak{o}_k/4\mathfrak{o}_k)^\times$, because

$$\frac{\varepsilon_\Omega(\alpha)^{\frac{1}{2}(N\beta-1)}}{\varepsilon_\Omega(\beta)^{\frac{1}{2}(N\alpha-1)}} = \frac{e\big(\frac{1}{8}(N\alpha-1)\big)^{\frac{1}{2}(N\beta-1)}\cdot\varepsilon_\Omega(\alpha)^{\frac{1}{2}(N\beta-1)}}{e\big(\frac{1}{8}(N\beta-1)\big)^{\frac{1}{2}(N\alpha-1)}\cdot\varepsilon_\Omega(\beta)^{\frac{1}{2}(N\alpha-1)}} = \frac{\tilde{\varepsilon}_\Omega(\alpha)^{\frac{1}{2}(N\beta-1)}}{\tilde{\varepsilon}_\Omega(\beta)^{\frac{1}{2}(N\alpha-1)}}.$$

In the case where d_k is odd, since ε_Ω is a character of order 2, we have

$$\frac{\varepsilon_\Omega(\alpha)^{\frac{1}{2}(N\beta-1)}}{\varepsilon_\Omega(\beta)^{\frac{1}{2}(N\alpha-1)}} = \varepsilon_\Omega(\alpha)^{\frac{1}{2}(N\beta-1)}\ \varepsilon_\Omega(\beta)^{\frac{1}{2}(N\alpha-1)}.$$

In the case where $d_k = 4d_0$ with $d_0 \equiv 1 \pmod 4$, $N\alpha \equiv 1 \pmod 4$ for any $\alpha \in J^*(2)$, and hence we have the same equality as above. In the remaining case where $d_k = 4d_0$ with $d_0 \equiv 2 \pmod 4$, since $\tilde{\varepsilon}_\Omega$ is a character of order 2, we have

$$\frac{\tilde{\varepsilon}_\Omega(\alpha)^{\frac{1}{2}(N\beta-1)}}{\tilde{\varepsilon}_\Omega(\beta)^{\frac{1}{2}(N\alpha-1)}} = \tilde{\varepsilon}_\Omega(\alpha)^{\frac{1}{2}(N\beta-1)}\ \tilde{\varepsilon}_\Omega(\beta)^{\frac{1}{2}(N\alpha-1)}.$$

Here we bring to mind that Hajir and Villegas ([5]) which proved the following law of quadratic reciprocity.

Theorem 4-2 ([5, Theorem 21]). *For $\alpha, \beta \in J^*(2)$ such that $(\alpha, \beta) = 1$,*

$$\left(\frac{\alpha}{\beta}\right)_2 \left(\frac{\beta}{\alpha}\right)_2 = (-1)^{\frac{1}{4}(N\alpha-1)(N\beta-1)}\ \kappa_4(\alpha)^{\frac{1}{2}(N\beta-1)}\kappa_4(\beta)^{\frac{1}{2}(N\alpha-1)}.$$

In Theorem 4-2, κ_4 is a certain character of $(\mathfrak{o}_k/4\mathfrak{o}_k)^\times$ defined with use of the Galois action on the quotients of η-values of Dedekind. For a precise explanation for κ_4, one should refer to [5]. Especially Lemma 12 in [5] is useful for the computation of $\kappa_4(\alpha)$.

Compairing Theorem 4-1 with Theorem 4-2, there arises some interesting questions as follows:

***Q*1.** What could be the relation between κ_4 and ε_Ω (or $\tilde{\varepsilon}_\Omega$) ?

***Q*2.** Does the definition of ε_Ω depend essentially on the $\mathfrak{o}_k$-ideal Ω ?

In the rest of this section, we attempt to make clear these questions. Hereafter we let $\{\omega,\ 1\}$ be the basis of $\mathfrak{o}_k$ and let ω be so normalized as follows:

$$\omega = \begin{cases} \dfrac{-1+\sqrt{-d_k}}{2} & \text{if } \ d_k \equiv 3 \pmod 4. \\[2ex] \sqrt{-d_0} & \text{if } \ d_k = 4d_0. \end{cases}$$

We take $\mathfrak{o}_k$ itself for Ω, and we write ε_1 for $\varepsilon_{\mathfrak{o}_k}$ defined using the basis $\{\omega,\ 1\}$. Then, from a numerical computation, we have the following

Theorem 4-3. (i) *In the case where d_k is odd, both of ε_1 and κ_4 are of order* 2*, and $\varepsilon_1 = \kappa_4$.*
(ii) *In the case where $d_k = 4d_0$ with $d_0 \equiv 1 \pmod 4$, both of ε_1 and κ_4 are of order* 4*, and $\varepsilon_1 = \kappa_4^3$.*
(iii) *In the case where $d_k = 4d_0$ with $d_0 \equiv 2 \pmod 4$, both of $\tilde{\varepsilon}_1$ and κ_4 are of order* 2*, and $\tilde{\varepsilon}_1 = \kappa_4$. Herein $\tilde{\varepsilon}_1$ means a character defined by $\tilde{\varepsilon}_1(\alpha) = e\big(\frac{1}{8}(N\alpha - 1)\big)\varepsilon_1(\alpha)$* for $\alpha \in J^*(2)$.

Now let Ω and Ω_1 be two $\mathfrak{o}_k$-ideals which are similar to each other, i.e. $\Omega_1 = \mu\Omega$ with some $\mu \in k$. We fix a basis $\{\omega_1, \omega_2\}$ of Ω and take $\{\mu\omega_1, \mu\omega_2\}$ for a basis of $\Omega_1 = \mu\Omega$. Then, from the definition (2.1) and the homogeneous property of $\mathcal{K}_\Omega(z)$, we see that

$$\varepsilon_{\Omega_1}(\alpha) = \varepsilon_{\mu\Omega}(\alpha) = \varepsilon_\Omega(\alpha) \quad \text{for any } \alpha \in J^*(2).$$

In each ideal class of k, there exists a prime ideal $\mathfrak{p}$ of degree 1 such that

$$N\mathfrak{p} = p \equiv \begin{cases} 1 \pmod 4 & \text{when } d_k = 4d_0 \text{ with } d_0 \equiv 1 \pmod 4. \\ 1 \pmod 2 & \text{otherwise.} \end{cases}$$

Namely, we know that

$$(\, w_{\mathcal{H}}, 4\,) = \begin{cases} 4 & \text{when } d_k = 4d_0 \text{ with } d_0 \equiv 1 \pmod 4, \\ 2 & \text{otherwise,} \end{cases}$$

where $w_{\mathcal{H}}$ means the number of roots of unity in the Hilbert class field $\mathcal{H}$ over k. We let $\{\,\omega + \nu\,,\, p\,\}$ be a canonical basis of $\mathfrak{p}$. ν is uniquely determined modulo p. Here we may assume additionally that

$$\nu \equiv \begin{cases} 0 \pmod 8 & \text{when } d_k \equiv 3 \pmod 4. \\ 1 \pmod 8 & \text{when } d_k \equiv 0 \pmod 4. \end{cases}$$

Then, by a numerical computation with use of the basis $\{\,\omega + \nu\,,\, p\,\}$ of $\mathfrak{p}$, we can confirm that $\varepsilon_{\mathfrak{p}}(\alpha) = \varepsilon_1(\alpha)$ for any $\alpha \in J^*(2)$. Consequently, we can summarize our arguments as follows:

Theorem 4-4. (i) *When $d_k = 4d_0$ with $d_0 \equiv 1 \pmod 4$, the definition of ε_Ω depends neither on the choice of basis of Ω nor on Ω itself, and $\varepsilon_\Omega = \varepsilon_1 = \kappa_4^3$.*
(ii) *When $d_k =$ odd or $d_k = 4d_0$ with $d_0 \equiv 2 \pmod 4$, for any Ω, ε_Ω is equal to one of $\{\,\varepsilon_1\,,\, \varepsilon_1 \cdot \chi_4 \circ N\,\}$*

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